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1986 J. Phys. A: Math. Gen. 19 L1003

(http://iopscience.iop.org/0305-4470/19/16/010)

J. Phys. A: Math. Gen. 19 (1986) L1003-L1008. Printed in Great Britain

LETTER TO THE EDITOR

Multivalley structure in Kauffman's model: analogy with spin glasses

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Received 18 July 1986

Abstract. Kauffman's model describes a random network of automata. Our calculations indicate that the multivalley structure of the basins of attraction in Kauffman's model is very similar to that of infinite-range spin glasses. The similarity with spin glasses is tested quantitatively by computing the probability that two initial configurations fall into the same valley.

Networks of random automata are simple random systems which exhibit interesting dynamical properties [1-3]. They have several qualitative features in common with other strongly disordered systems like spin glasses: randomness, the role played by the overlaps between different configurations and the phase space split into many valleys. In this letter we study quantitatively how phase space is broken into basins of attraction and compare this multivalley structure with what is known in the mean-field theory of spin glasses [4-6].

We consider here Kauffman's model [7-13]. The system consists of N sites, each site containing an Ising spin ($\sigma_i = 1$ or 0). The time evolution is determined by N functions f_i independently chosen for each site i and by the choice of K input sites $j_1(i), j_2(i), \ldots, j_K(i)$ for each site i (K is a parameter of the model). Each function f_i is specified once its 2^K possible values are chosen. In Kauffman's model, which is an infinite-dimension model [14], the input sites $j_1(i) \ldots j_K(i)$ are chosen randomly among the N sites and each value of the 2^K possible values of each function f_i is chosen to be 1 with probability $\frac{1}{2}$ or 0 with probability $\frac{1}{2}$.

The functions f_i and the input sites $j_1(i) \ldots j_K(i)$ do not change with time (the disorder is quenched) and the time evolution of the spin configurations is given by

$$\sigma_i(t+1) = f_i(\sigma_{j_1}(t), \ldots, \sigma_{j_K}(t)). \tag{1}$$

Since the system is deterministic and has a finite number (2^N) of configurations, the evolution of any configuration ends up by being periodic. Thus all attractors are cycles.

A given sample (defined by its set of functions f_i and input sites $j_1(i) \ldots k_K(i)$) has a certain number of attractors. The number S of these attractors, their periods and their basins of attraction are, of course, sample dependent. For each sample of N sites, let us call Ω_s the number of initial spin configurations which fall onto the sth attractor.

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0305-4470/86/161003+06$02.50

Since each spin configuration belongs to a single basin of attraction, one has

$$\sum_s \Omega_s = 2^N. \tag{2}$$

We can then define the weight W_s of the sth attractor by

$$W_s = \Omega_s / 2^N. \tag{3}$$

W_s is just the normalised size of the sth basin of attraction, i.e. W_s is the probability that a randomly chosen configuration at time $t=0$ will fall onto the sth attractor.

For finite N, the number S of attractors and their weights W_s change from sample to sample. One can then wonder what happens in the thermodynamic limit ($N \to \infty$).

Is phase space divided into more and more basins of attraction as $N \to \infty$?

Do all the weights W_s become smaller and smaller as N increases or do a few of them remain finite?

Do the sizes of the biggest basins of attraction fluctuate from sample to sample when $N \to \infty$?

All these questions have already been posed in the theory of spin glasses where it was shown that in the mean-field cases (the Sherrington-Kirkpatrick model and random energy model) [4-6] the phase space is divided into an infinite number of valleys but the valleys with the biggest weights have a finite weight and these weights fluctuate from sample to sample in the limit $N \to \infty$.

In order to see whether such effects are present in Kauffman's model, an easy quantity to consider is

$$Y = \sum_s W_s^2. \tag{4}$$

If $Y \to 0$ as $N \to \infty$, this means that, in the thermodynamic limit, all the weights become smaller and smaller. On the other hand, if Y remains finite, there are a few big basins of attraction which fill almost the whole phase space. Also if Y fluctuates from sample to sample, this means that the sizes of the biggest basins of attraction fluctuate. So we see that, to understand the multivalley structure of phase space, it is useful to compute $\bar{Y}$ and $\overline{Y^2}$ (where the average is done over disorder, i.e. over different choices of the functions f_i and of the input sites $j_1(i) \dots j_K(i)$).

To compute $\bar{Y}$ and $\overline{Y^2}$, we have used two different methods.

For small sizes, $N \leq 14$, we have computed all the basins of attraction and their sizes for each sample. So we could obtain Y for each sample by using (4). Then we averaged over 10^4 samples. The computer time needed in this first method increases as 2^N since we have to visit each spin configuration at least once. In this first approach, each sample is built randomly but then Y for this sample is computed exactly.

For larger sizes, $N > 14$, we used a completely different approach: a stochastic one. For each sample, we choose two random configurations of the spins. Then we iterate these two configurations until they fall on their attractors and then we check if they fall onto the same attractor. The probability that two randomly chosen initial spin configurations fall onto the same attractor is Y for a given sample. (This can be understood easily. W_s is the probability that a random initial configuration falls on attractor s and W_s^2 is the probability that two random initial configurations fall onto the sth attractor.) So to measure $\bar{Y}$ we iterated only two configurations for each sample and we averaged the number of times they fall onto the same attractor over many samples (10^4 samples). To measure $\overline{Y^2}$, we iterated for each sample four configurations (say, configurations A, B, C and D) and $\overline{Y^2}$ is just the average over many samples that A and B fall on the same attractor α and that C and D fall on the same attractor β (α and β need not be the same). With this second approach the computer time increases like $NT(N)$ where $T(N)$ is the typical time for a configuration to fall onto an attractor and to complete one cycle. It turns out that $T(N)$ depends on K [8] and for $K=1,2$ we could increase N up to 224 whereas we could only go up to $n=40$ for $K=3$ and $N=28$ for $K=4$, because $T(N)$ increases much faster with N.

Since to compute $\overline{Y^2}$ we need to iterate four spin configurations, we could also measure Y_3 and Y_4 defined by

$$Y_3=\sum_s W_s^3 \tag{5}$$

$$Y_4=\sum_s W_s^4. \tag{6}$$

$\overline{Y_3}$ is the average number of times that A, B and C fall on the same attractor and $\overline{Y_4}$ is the average number of times that the four configurations fall on the same attractor.

All these quantities can be computed in the mean-field theory of spin glasses either directly in the random energy model [6] or using the independent random free energy picture for the Sherrington–Kirkpatrick model [5] and one finds that for spin glasses they are all related:

$$\overline{Y^2}=\tfrac{1}{3}(\bar{Y}+2\bar{Y}^2) \tag{7}$$

$$\overline{Y_3}=\tfrac{1}{2}\bar{Y}(1+\bar{Y}) \tag{8}$$

$$\overline{Y_4}=\tfrac{1}{6}\bar{Y}(1+\bar{Y})(2+\bar{Y}). \tag{9}$$

All these formulae can easily be obtained from formulae (10), (11), (31) and (32) of Derrida and Toulouse [6].

In figure 1(a), we show $\bar{Y}$ measured for several sizes for $K=1$, 2, 3 and 4. We see that for $k=1,3$ and 4, $\bar{Y}$ seems to have a finite limit as $N\to\infty$, whereas for $K=2$, $\bar{Y}$ decreases slowly with N (notice that the horizontal scale is logarithmic). Therefore our results indicate that, at least for $K\neq 2$, there remain in the thermodynamic limit attractors which have a finite weight.

In figure 1(b), we show the N dependence of $\overline{Y^2}-(\bar{Y})^2$. Within our error bars, this quantity does not seem to vanish as $N\to\infty$, indicating that Y remains sample dependent even in the thermodynamic limit. Even in the cases $K=3$ and $K=4$ where the largest size value is lower than previous values, the error bar is too large to conclude that $\overline{Y^2}-(\bar{Y})^2$ decreases with N.

One should notice that for $K=2$, if $\bar{Y}\to 0$ as $N\to\infty$, then $\overline{Y^2}-(\bar{Y})^2$ must also vanish as $N\to\infty$ because Y is always positive. We find our results of figures 2(a) and (b) rather contradictory because figure 1(a) indicates that $\bar{Y}\to 0$ and figure 1(b) indicates that $\overline{Y^2}-(\bar{Y})^2$ remains finite. The fact that $K=2$ is more difficult to analyse should not be surprising because $K=2$ is a marginal case where the convergence to the thermodynamic limit is slower [13].

One should also notice that for Kauffman's model, even for finite N, phase space can be broken into several valleys. This is, of course, a difference with spin glasses where it is only in the thermodynamic limit that the multivalley structure appears.

So we see that figures 1(a) and (b) show that the way phase space is broken into valleys for Kauffman's model is very similar to what it is for infinite-range spin glass models (in the thermodynamic limit).

In order to test whether this analogy is deeper, we tried to see whether formulae (7), (8) and (9) could be valid in Kauffman's model. In figures 2(a), (b) and (c), we have plotted $\overline{Y^2}$, $\overline{Y_3}$ and $\overline{Y_4}$ against $\bar{Y}$.

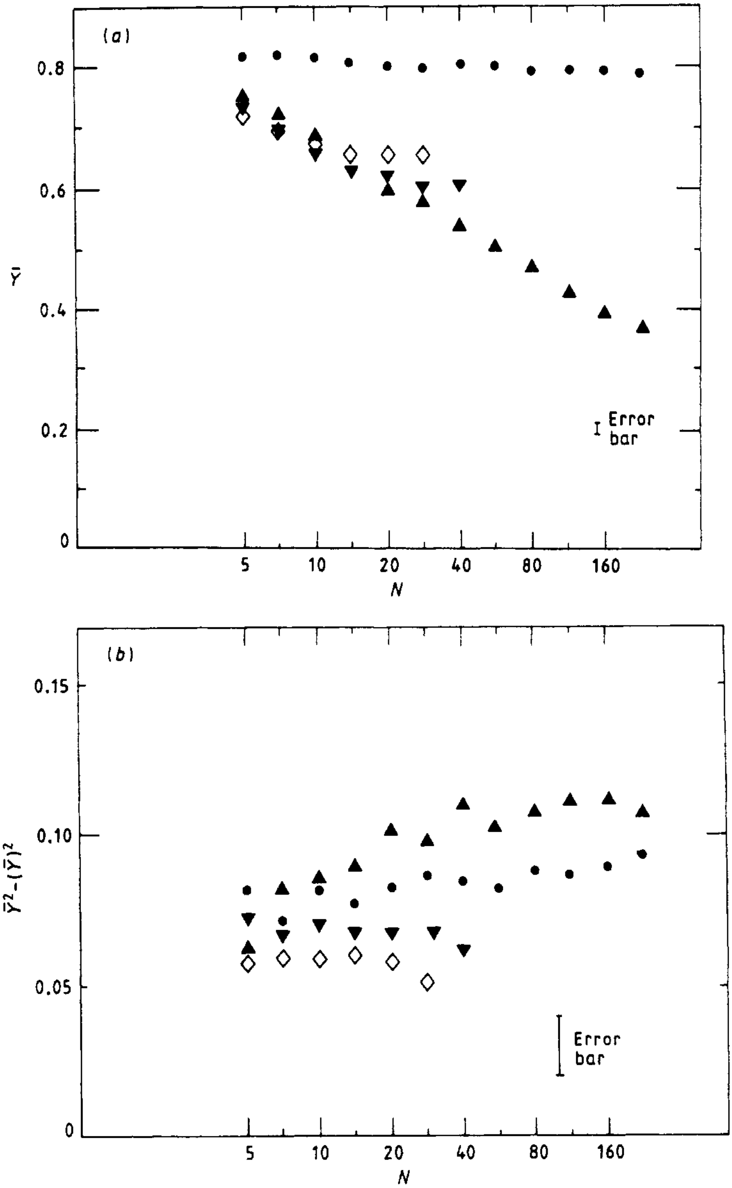
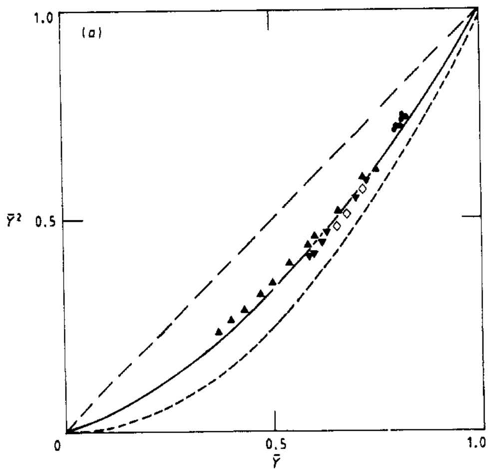
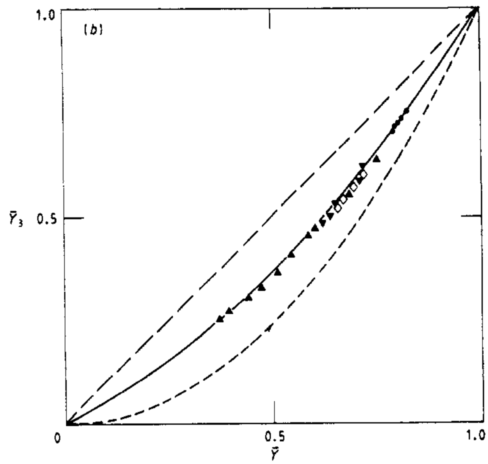


Figure 1. (a) $\bar{Y}$ plotted against the number of spins N for $K = 1$(●), 2(▲), 3(▼) and 4(◇). Each point represents an average over 10 000 samples. (b) $\overline{Y^2} - (\bar{Y})^2$ plotted against N.

The full curves represent formulae (7), (8) and (9). We see that our numerical data seem to agree rather well with these expressions even for finite N.

The agreement between our data points and the curves in figure 2 should, however, be interpreted with the following precaution: Y is a variable which is always between 0 and 1. Therefore, one has $(\bar{Y})^2 < \overline{Y^2} < \bar{Y}$. The broken curves in figure 2(a) represent these bounds. Hence, even if formula (7) is not true for Kauffman's model, our data points must fall within these bounds. (One can also find bounds for $\overline{Y_3}$ and $\overline{Y_4}$:





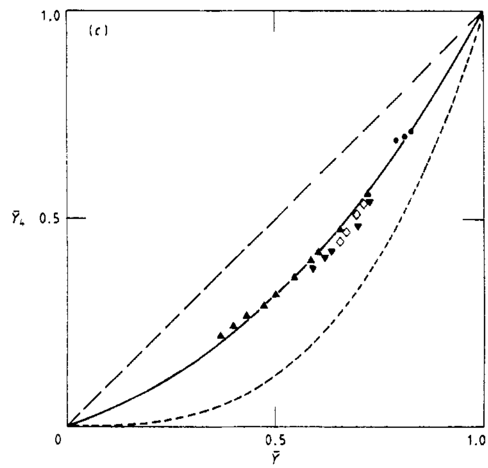


Figure 2. (a) $\overline{Y^2}$ plotted against $\bar{Y}$ for (●) $K=1$ and $5 \leqslant N \leqslant 160$, (▲) $K=2$ and $5 \leqslant N \leqslant 160$, (▼) $K=3$ and $5 \leqslant N \leqslant 40$, (◇) $K=4$ and $5 \leqslant N \leqslant 28$. Each point represents the average over 10 000 samples. The full curve represents formula (7): $\overline{Y^2} = \frac{1}{3}(\bar{Y} + 2\bar{Y}^2)$, whereas the broken curves represent the bounds $\bar{Y}^2 \leqslant \overline{Y^2} \leqslant \bar{Y}$. ($b$) $\overline{Y_3}$ plotted against $\bar{Y}$. The full curve represents formula (8): $\overline{Y_3} = \frac{1}{2}\bar{Y}(1+\bar{Y})$, whereas the bounds are $\bar{Y}^2 \leqslant \overline{Y_3} \leqslant \bar{Y}$. ($c$) $\overline{Y_4}$ plotted against $\bar{Y}$. The full curve shows formula (9): $\overline{Y_4} = \frac{1}{6}\bar{Y}(1+\bar{Y})(2+\bar{Y})$ whereas the bounds are $\bar{Y}^3 \leqslant \overline{Y_4} \leqslant \bar{Y}$.

$(\bar{Y})^2 < \overline{Y_3} < \bar{Y}$ and $(\bar{Y})^3 < \overline{Y_4} < \bar{Y}$. These bounds are again the broken curves of figures 2(b) and (c).)

In Kauffman's model, one expects a different behaviour for $K < 2$ and $K > 2$ [7, 8, 12, 13]. Our results presented here do not show any qualitative difference for $K=1$ and $K=3$ and 4. Just at $K=2$, the limit $N \to \infty$ is reached more slowly.

In conclusion, our calculations indicate that the multivalley structure in Kauffman's model is rather similar to what it is in infinite-range spin glasses, even quantitatively and for finite N. Very large valleys exist which take up a finite fraction of phase space. It would of course, be interesting to compute analytically $\bar{Y}$, $\overline{Y^2}$, $\overline{Y_3}$ and $\overline{Y_4}$ or to improve the statistics and to study larger sizes in order to see whether formulae (7), (8) and (9) are exactly or just approximately true. Also it would be interesting to compute higher moments of Y in order to see whether the whole probability distribution of Y is universal (i.e. whether it is known when the mean $\bar{Y}$ is known) as it is for infinite-range spin glasses [4–6].

It is possible in Kauffman's model to define $Y(q)$, as in spin glasses. It is defined by the probability that two initial configurations which have an overlap q fall onto the same attractor. It would be interesting to see whether this $Y(q)$ has the same statistical properties as the Y which was studied in the present letter.

Another important question is to know whether the results described here remain true in finite dimension. One can define Kauffman's model in finite dimension by putting the spins on a lattice [14]. We did some preliminary calculations on the square lattice ($K=4$) and we found that Y decreases very quickly to zero as one increases the size of the system. Therefore the multivalley structure of Kauffman's model is probably very different in finite dimension, again similar to spin glasses.

We would like to thank H J Herrmann, D Stauffer and G Weisbuch for useful discussions, and Professor H B Nielsen and the Danish Council for Research Policy and Planning for economic support.

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